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J. Phys. A: Math. Gen. **15** (1982) 1849–1858. Printed in Great Britain

Diode–resistor percolation in two and three dimensions: I. Upper bounds on critical probability

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Received 20 November 1981

Abstract. We obtain upper bounds to the critical probability for percolation in a random network made of oriented diodes and resistors. It is shown that for the square lattice $p_c < 0.3700$ and for the simple cubic lattice $p_c < 0.2417$.

1. Introduction and outline

Although the directed percolation problem was formulated by Broadbent and Hammersley along with undirected percolation in their classic paper in 1957, it has not been studied very much until recently. Besides their obvious relevance to many physical situations (fluid flow through porous media under gravity, hopping conduction in strong electric fields, spread of forest fires in the presence of wind, etc; see Obukhov (1980) and Van Lien and Shklovskii (1981)), directed percolation problems are interesting because they show many qualitatively new features usually not evident in undirected percolation. The percolation problem of the determination of the properties of a random network of diodes and insulators was studied by Blease (1977a, b, c), who obtained estimates for the critical percolation probabilities and critical exponents for a large number of lattices from series expansions, and showed that the problem lies in a different universality class from undirected percolation. Obukhov (1980) showed that the upper critical dimension for directed percolation is 5, and studied the structure of the infinite cluster in $5-\varepsilon$ dimensions. Cardy and Sugar (1980) have shown that the directed percolation problem is equivalent to Reggeon field theory, which models the behaviour of scattering cross sections of elementary particles at high energies.

The problem has been studied using Monte Carlo simulations (Kertesz and Vicsek 1980. Dhar and Barma 1981) and renormalisation group techniques (Kinzel and Yeomans 1981, Phani and Dhar 1982), and its critical exponents determined. Domany and Kinzel (1981) have studied this problem on a square lattice in the case of unequal horizontal and vertical bond probabilities, and noticed that the problem can be solved exactly in the case when one of these two bond probabilities is exactly 1.

More recently, problems with a random mixture of no-way, one-way and two-way bonds have been studied. In electrical terminology these may be called diode–resistor–insulator percolation problems. Reynolds (1981) and Redner (1982, Redner and Brown 1981) allow the diode orientations to be arbitrary, while Dhar *et al* (1981) have considered mainly the case when all diodes are oriented in the direction of the positive axes.

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In this paper we study the diode–resistor percolation problem in two and three dimensions. The problem is defined by the requirement that each of the bonds of the lattice can be one-way conducting or two-way conducting, with probabilities q and p respectively, independently of other bonds ($p+q=1$). The orientation of diodes is pre-assigned to be in the direction of increasing x, y or z coordinates and is not a random variable.

If the concentration of resistors p is less than a critical value p_c, a point source of fluid cannot wet all points of the lattice. The wetted region, far away from the source, is confined to a convex region with less than half of the sites of the plane wet. As p is increased above p_c, there is an instability and the entire plane is wetted with probability 1. The fractional number of wetted sites jumps discontinuously from $\frac{1}{2}$ to 1 as p is increased above p_c.

The diode–resistor percolation is the simplest non-trivial model for studying the directional effects in percolation problems. In two dimensions it is dual to the diode–insulator percolation problem, and this property can be used to advantage as some physical quantities are easier to study in the diode–resistor problem than in the dual diode–insulator percolation. In an earlier paper we used this property to determine the variation of the wedge angle in two dimensions using Monte Carlo simulations (Dhar *et al* 1981).

The organisation of this paper is as follows. In § 2 we describe a technique to determine upper bounds to the critical probability p_c on a square lattice. The convergence of our technique is considerably better than earlier techniques, and its asymptotic convergence is discussed briefly. In § 3 we apply the same technique to determine upper bounds to the critical probability for a simple cubic lattice. Some algebraic details of the derivation of third-order bounds to the critical probability for the square lattice are given in an Appendix.

2. Upper bounds to the critical probability in diode–resistor percolation on a square lattice

In this section we describe a technique to derive a monotonically decreasing sequence of rigorous upper bounds to the critical probability in the diode–resistor percolation problem by systematically taking into account more and more backflow paths. We also use the technique to obtain a series expansion for the tangent of the half-wedge angle $\theta(p)$ in powers of p.

The technique may be summarised as follows. Starting at the origin, we determine a sequence of points P_i, $i = 0 \ldots \infty$, such that the x ordinate of P_i is $-i$ and a source at P_i would wet P_{i+1}. The exact choice of P_{i+1}, given P_i, depends on the configuration of bonds in the neighbourhood of P_i. The sequence $\{P_i\}$ is defined so that the ordinates define a Markov stochastic process (with finite memory). We use this Markovian property to determine the average direction in which the points P_i lie with respect to the origin. By construction, a source at the origin wets all the points P_i. The average direction of P_i thus defined gives us a lower bound to the wedge angle in the diode–resistor percolation. The bounds on the critical probability follow from the requirement that, for probabilities less than the critical probability, the wedge angle be less than π.

As an example of the wetting strategies (decision rules determining P_{i+1}, given P_i), we describe below a strategy, to be called the k strategy, in which in determining

P_{i+1} from P_i we consider only backflow paths involving not more than k adjacent columns of vertical bonds to the right of and including the column $x=-i-1$ and the intervening horizontal bonds.

Let V_i be the set of vertical bonds in the column $x=-i$, and let H_i be the set of horizontal bonds between the columns $x=-i$ and $x=-i-1$. We define P_0 to be the lowest point $(0, y_0)$ in the column $x=0$ which may be wetted by a source at the origin using bonds lying in $V_0 \cup H_{-1} \cup V_{-1}$ only. (Here $\cup$ denotes set-theoretical union.) Let the coordinates of the point P_i be $(-i, y_i)$ (figure 1). Let $(-i-1, y_i+r_{i+1})$ be the coordinate of the lowest wetted point P_i' in the column $x=-i-1$ if the source is at P_i, and using wetting paths lying in $V_i \cup H_i$. Let $y_i+r_{i+1}-s_{i+1}$ be the ordinate of the lowest wetted point P_i'' in the column $x=-i-1$ if the source is at P_i' and wetting paths lie completely in V_{i+1}. We then determine the point $P_i''' \equiv (-i-1, y_i+r_{i+1}-s_{i+1}-t_{i+1})$, the lowest point in its column wetted from a source at P_i'' using wetting paths in $V_{i+1} \cup H_i \cup V_i$. Finally, we determine the point $P_{i+1} \equiv (-i-1, y_i+r_{i+1}-s_{i+1}-t_{i+1}-u_{i+1})$ which is defined as the lowest wetted point in the column $x=-i-1$ if the source is at P_i''', and wetting occurs only through bonds in $V_{i+1} \cup H_i \cup V_i \cup H_{i-1} \cup V_{i-1}$.

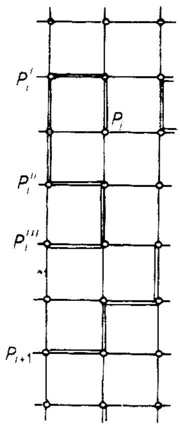

Figure 1. The strategy $k=3$. The single and double lines denote one-way and two-way bonds respectively. The positions of the points P_i', P_i'', P_i''' and P_{i+1} are shown for a specific configuration of diodes and resistors.

The above prescription determines the points $\{P_i\}$ recursively, and defines the wetting strategy for $k=3$. The strategies for other values of k are defined similarly. In the $k=0, 1, 2$ strategies the point P_{i+1} is identified with the points P_i', P_i'' and P_i''' respectively. In general, in the strategy k, P_{i+1} is defined as the lowest wetted point in the column $x=-i-1$ if the source is at P_i, and wetting occurs using bonds lying in H_j or V_l $(i+1 \geq l > i+1-k,\ i > j > i+1-k)$.

The break-up of $y_{i+1}-y_i$ into four variables r_{i+1}, s_{i+1}, t_{i+1} and u_{i+1} is motivated by the observation that s_{i+1}, t_{i+1} and u_{i+1} may be called the wettings due to first-, second- and third-order backflows respectively. (In general, $y_{i+1}-y_i$ is broken into $k+1$ terms including up to the kth-order backflow term.) The distribution of the kth-order backflow variable depends only on earlier (lower i) variables of lower order, and we can determine these distributions successively.

Coming back to the special case $k=3$, we note that the determination of r_{i+1}, s_{i+1}, t_{i+1} and u_{i+1} only requires a knowledge of the bond configurations in V_{i+1}, H_i, V_i, H_{i-1} and V_{i-1}. Thus the sequence $\{r_i, s_i, t_i, u_i\}$ treated as a vector stochastic process is a Markov process with finite memory. A little reflection shows that this process is actually a simple Markov process, and the probability distribution of $(r_{i+1}, s_{i+1}, t_{i+1}, u_{i+1})$ depends only on (r_i, s_i, t_i, u_i) and not on earlier values of r, s, t or u. Symbolically we write

$$\Pr(\{r_i, s_i, t_i, u_i\}) = \Pr(r_0, s_0, t_0, u_0) \prod_i \Pr(r_i, s_i, t_i, u_i | r_{i-1}, s_{i-1}, t_{i-1}, u_{i-1})$$

where $\Pr(E)$ denotes the probability of an event E, and $\Pr(E|F)$ denotes the conditional probability of the event E, given F.

In the Appendix we have described the transition matrix of this process in some detail and calculated the mean values of r_i, s_i, t_i and u_i. These values are independent of i. The expectation value of y_i is clearly given by

$$\langle y_i \rangle = \langle y_0 \rangle + i(\langle r \rangle - \langle s \rangle - \langle t \rangle - \langle u \rangle). \tag{1}$$

The average direction of motion ϕ, using this strategy of wetting with respect to the direction $x = y > 0$, is given by

$$\cot(\phi - \pi/4) = \langle r \rangle - \langle s \rangle - \langle t \rangle - \langle u \rangle. \tag{2}$$

Numerical evaluation of the right-hand side of equation (2) using the explicit functional form derived in the Appendix shows that it is less than 1 if $p > 0.3702$. Since for all $p < p_c$ the wetted cluster is confined to a wedge of half-wedge angle less than $\pi/2$, this implies that

$$p_c(\text{square lattice}) < 0.3702. \tag{3}$$

The corresponding bounds for the $k = 0$, 1, 2 strategies are 0.5, 0.3820 and 0.3739 respectively. These bounds can of course be improved by calculating higher-order backflow terms. The contribution of kth-order backflows is easily seen to be of order p^{2k}. For example, if $\{v_i\}$ denote the fourth-order backflow terms in a $k > 4$ strategy, it can be shown that

$$\langle v_i \rangle = p^8 + O(p^9). \tag{4}$$

Using the above equation we can write a series expansion for the half-wedge angle $\theta(p)$:

$$\cot[\theta(p) - \pi/4] = \langle r \rangle - \langle s \rangle - \langle t \rangle - \langle u \rangle - p^8 - O(p^9). \tag{5}$$

Using the expressions for $\langle r \rangle$, $\langle s \rangle$, $\langle t \rangle$ and $\langle u \rangle$ given in the Appendix, this equation may be re-expressed as

$$\tan[\theta(p) - \pi/4] = p + p^2 + 2p^3 + 4p^4 + 8p^5 + 17p^6 + 38p^7 + 85p^8 + 193p^9 + 450p^{10} + O(p^{11}). \tag{6}$$

The upper bounds on the critical probability of the diode–resistor percolation (p_c^{DRP}) on a square lattice can be translated into lower bounds on the critical probability for the diode–insulator percolation (p_c^{DIP}) on a square lattice as these two are related to each other by the duality relation (Dhar *et al* 1981)

$$p_c^{\text{DIP}} + p_c^{\text{DRP}} = 1. \tag{7}$$

Equation (3) then implies that

$$p_c^{\text{DIP}} > 0.6298. \tag{8}$$

It is interesting to compare our lower bounds on p_c^{DIP} thus derived with earlier known lower bounds. Mauldon (1961) obtained a monotonic increasing sequence of bounds to p_c^{DIP} by considering Markov processes simpler than the diode–insulator percolation, and converging to it. His first lower bound was $p_c^{\mathrm{DIP}} > 0.5858$, and his best bound (eight terms in his sequence) was 0.6199. Blease (1977b) used the result (due to Hammersley) that for $p > p_c^{\mathrm{DIP}}$ the expected number of sites wetted by a point source and at a distance n from it must be greater than 1. For $n = 27$ his result was $p_c^{\mathrm{DIP}} > 0.5895$, while for $n = 15$ he obtained $p_c^{\mathrm{DIP}} > 0.5758$. These results should be compared with the true value of p_c estimated by extrapolations of series expansions $p_c^{\mathrm{DIP}} = 0.6446 \pm 0.0002$. Clearly the convergence of these bounds to the true value is very slow.

Recently Gray *et al* (1980) have shown that $p_c^{\mathrm{DIP}} > 0.6231$ by considering bounds on the average inclinations of 'ceilings' and 'floors' (these are defined as boundaries outside which no sites are wet), a method which is quite similar in spirit to ours. We note, however, that our $k = 3$ lower bound (0.6298) is somewhat better than their value (0.6231) using one-step ceilings.

These bounds can be improved by taking into account backflows of higher and higher order. In the limiting case $k \to \infty$, clearly we get the correct critical probability. However, the small improvement in the $k = 3$ bound over the $k = 2$ value suggests that the convergence of these bounds to a true critical value is algebraic and not exponential. For example, assuming the two-point correlation function $G(R)$ below p_c^{DIP} varies as $G(R) \sim R^{\eta - z/2} \exp(-\varepsilon^{\nu} R)$ (Cardy and Sugar 1980), it is easy to see that the difference between p_c^{DIP} and the bound calculated using the method of Blease and determining the correlations up to a distance n varies as $[(\log n)/n]^{1/\nu}$. On the other hand, this difference is $\mathrm{O}(1/n)$ if we determine a bound $p_{c,n}$ on p_c^{DIP} by requiring that the sum of the first n terms on the right-hand of equation (6) be 1. (We assume that coefficients in the expansion vary as $p_c^{-n} n^{-1-b}$ for large n.) Since a calculation of kth-order backflows is somewhat better than determining the first $2k+4$ terms in the Taylor expansion of $\tan[\theta(p) - \pi/4]$, the convergence of these bounds is at least $\mathrm{O}(1/n)$. The value of ν is about 1.73 and 1.27 in two and three dimensions respectively, and the calculation of correlations up to a distance n requires steps of order $\exp(n^2)$ and $\exp(n^3)$ respectively; the gain in convergence is substantial using our technique. The slow convergence of rigorous bounds to critical parameters is also encountered in other problems, and reflects the divergence of the correlation length near criticality.

The exact calculation of higher-order backflows is quite difficult. An easier approach (though less systematic) is to obtain lower-bound estimates of their contributions valid for all k. For example, the simple observation that the contribution of the kth-order backflow term to $\tan[\theta(p) - \pi/4]$ is not less than $p^{2k}(1-pq)^{2-2k}$ for $k > 1$ may be used to obtain a slightly stronger result $p_c^{\mathrm{DRP}} < 0.3700$. Estimates which are sharper than above can be obtained with some effort, but will not be discussed here.

3. Upper bounds to the critical probability for diode–resistor percolation on the cubic lattice

Upper bounds to the critical probability for the diode–resistor percolation on the simple cubic lattice are derived similarly. To avoid tedious algebra, we illustrate the technique using the simplest $k = 1$ wetting strategy in which no x or y bond is ever traversed in the positive direction (we assume that all diodes are oriented in the direction of increasing x, y or z coordinates). The wetting strategy is defined as follows.

(i) Start at the origin. Choose $i = 0$.

(ii) Move in the positive z direction until the first allowed step in the negative x or negative y directions is encountered. Let the number of steps taken be r_i.

(iii) (a) If only one of the negative x or negative y steps is allowed, take that step, and then move in the negative z direction until the first one-way z bond is encountered. Let the number of z steps taken be s_i.
(b) If both negative x and negative y steps are allowed, choose the one corresponding to the larger value of s_i.

(iv) Increase i by 1, and go back to (ii).

It is easy to see that, using this strategy, no bond is ever tested twice. The variables $\{r_i, s_i\}$ are mutually independent random variables with distributions given by

$$\Pr(r_i = r) = q^{2r}(1-q^2) \tag{9}$$

$$\Pr(s_i = s) = \frac{2pq}{1-q^2} p^s q + \frac{p^2}{1-q^2} p^s q(2-2p^s + p^s q). \tag{10}$$

The mean values of r_i and s_i are given by

$$\langle r_i \rangle = q^2/(1-q^2) \tag{11}$$

$$\langle s_i \rangle = (2p + 2p^2 - p^3)(2-p)^{-1}(1-p^2)^{-1}. \tag{12}$$

Numerical evaluation shows that $\langle r_i \rangle - \langle s_i \rangle$ is less than 1 if $p > 0.2417$. But this would imply that the average direction of motion using this strategy makes an obtuse angle with the direction $x = y = z > 0$. But such a situation is clearly unstable, and implies that all sites of the lattice are wetted by a point source. We conclude that

$$p_c^{\mathrm{DRP}}(\text{simple cubic}) < 0.2417. \tag{13}$$

Acknowledgments

I would like to thank Drs M Barma and M K Phani for useful discussions, and Professor W Kinzel for pointing out an error in the arguments in an earlier version of the manuscript.

Appendix

In this Appendix we calculate the mean values $\langle r_i \rangle$, $\langle s_i \rangle$, $\langle t_i \rangle$ and $\langle u_i \rangle$ in the $k = 3$ backflow strategy. The arguments are not very straightforward, involving several steps. We outline them below in sufficient detail to enable the reader to reconstruct them without undue effort.

We note that the transition matrix of the random (vector) sequence $\{r_i, s_i, t_i, u_i\}$ can be factorised in the form

$$\begin{aligned}\Pr(r_i, s_i, t_i, u_i | r_{i-1}, s_{i-1}, t_{i-1}, u_{i-1}) \\ = \Pr(r_i) \Pr(s_i) \Pr(t_i | r_i s_i) \Pr(u_i | r_i, s_i, t_i, r_{i-1}, s_{i-1}).\end{aligned} \tag{A1}$$

This may be seen as follows. r_i is the number of vertical bonds that must be traversed upwards from p_i before the first allowed leftward step is encountered. This depends only on the bond configuration in H_i. Similarly, s_i depends only on the bond configuration in V_i. From the mutual independence of occupation probabilities for different bonds, it follows that $\{r_i\}$ and $\{s_i\}$ are mutually independent random variables. The probability distributions of $\{r_i\}$ and $\{s_i\}$ are geometrical distributions given by

$$\Pr(r_i = r) = pq^r, \qquad r = 0, 1, 2, \ldots \tag{A2}$$

$$\Pr(s_i = s) = qp^s, \qquad s = 0, 1, 2, \ldots. \tag{A3}$$

Mean values of r_i and s_i are easily calculated from these distributions and are given by

$$\langle r_i \rangle = q/p \tag{A4}$$

$$\langle s_i \rangle = p/q. \tag{A5}$$

The probability distribution of t_i depends on r_i and s_i because it depends on bonds in H_i, V_i and V_{i-1}, and some information about these bonds is contained in r_i and s_i. For example, direct downward wetting from the points P_{i-1} or $(-i, y_{i-1}+r_i-s_i)$ is not allowed. Also, there is no direct leftward wetting from $(-i+1, y_{i-1}+\Delta)$ for $0 \leqslant \Delta \leqslant r_i - 1$. If $r_i \geqslant s_i$, we must have $t_i = 0$ as no wetting paths giving $t_i \neq 0$ exist consistent with the above information. On the other hand, if $r_i < s_i$, additional wetting may occur with finite probability as wetting paths first going right to the column $x = -i+1$, then down and then left to $x = -i$ may exist.

If $r_i < s_i$, the conditional probability distribution of t_i is independent of the precise values of r_i and s_i, as the wetting paths involve bonds only to the right of and below P''_{i-1}. The only information about these bonds is that the bond directly below P_{i-1} is a diode. No other bonds would have been tested earlier according to our strategy. Let the conditional probability that $t_i = t$ given that $r_i < s_i$ be f_t, or more formally

$$\Pr(t_i = t | r_i, s_i) = \begin{cases} \delta_{t0} & \text{if } r_i \geqslant s_i \qquad \text{(A6}a\text{)} \\ f_t & \text{if } r_i < s_i. \qquad \text{(A6}b\text{)} \end{cases}$$

That u_i depends on r_i, s_i, t_i, r_{i-1} and s_{i-1} but not on t_{i-1} and u_{i-1} is based on a similar observation. Given $r_i < s_i$, the wetting paths contributing to non-zero t_i may be enumerated as follows. P''_{i-1} automatically wets the site $(-i+1, y_{i-1}+r_i-s_i)$. Let us say that the highest two-way flow allowing bonds in H_i having ordinate below $y_{i-1}+r_i-s_i$ is at $y_{i-1}+r_i-s_i-a$. Here $a \geqslant 1$. The probability distribution of a is geometrical

$$\Pr(a) = pq^{a-1}. \tag{A7}$$

In order that $t_i \neq 0$, the vertical bonds between $(-i+1, y_{i-1}+r_i-s_i)$ and $(-i+1, y_{i-1}+r_i-s_i-a)$ must all be resistors. The probability of this event is p^a. Let $(-i, y_i+r_i+s_i-a-b)$ be the lowest wetted point in the column $x = -i$ using only bonds in V_i if the source is at $(-i, y_i+r_i-s_i-a)$. Clearly, b takes values from 0 to ∞ with probabilities qp^b. Now let $(-i, y_i+r_i-s_i-a-b-c)$ be the lowest wetted point in the column $x = -i$ using bonds in H_{-i}, V_{-i} and V_{-i+1} and the source at $(-i, y_i+r_i-s_i-a-b)$. The same point would be wetted if the source were at $(-i, y_i+r_i-s_i)$. Hence we have $t_i = a+b+c$. The conditional probability distribution of c is clearly the same as that of t. We thus have

$$f_t = \sum_{a=1}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} pq^{q-1} p^a qp^b f_c \delta_{t,a+b+c} \qquad \text{for } t > 0. \tag{A8}$$

This equation can be solved by the method of generating functions. Define

$$F(x) = \sum_{t=0}^{\infty} f_t x^t. \tag{A9}$$

It is easily seen that the solution of equation (A8) is

$$F(x)=q(1-pqx)(1-px)(1-pq)^{-1}(1-xp-xpq+x^2p^2q-xp^2q)^{-1}. \tag{A10}$$

From equations (A6), (A9) and (A10) it follows immediately that

$$\langle t_i\rangle=\frac{p^2}{1-pq}\left.\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right|_{x=1}=\frac{p^4(1-p^2q)}{q^2(1-pq)^2}. \tag{A11}$$

We now determine $\langle u_i\rangle$. As noted earlier, the conditional probability distribution of u_i depends only on r_i, s_i, t_i, r_{i-1} and s_{i-1}. We write

$$\langle u_i\rangle=\sum u_i\,\mathrm{Pr}(r_i, s_i, t_i, r_{i-1}, s_{i-1}) \tag{A12}$$

where the summation extends over all possible values of r_i, s_i, t_i, u_i, r_{i-1} and s_{i-1}. If $r_i\geqslant s_i$, the argument used previously implies that u_i must be zero. Thus the summation in equation (A12) may be restricted to $r_i<s_i$.

Two cases arise. If $r_{i-1}<s_{i-1}$, then the point P'''_{i-1} automatically wets the point $(-i+2,\ y_{i-1}+r_i-s_i-t_i)$. In this case, the conditional distribution of u_i (given r_i, s_i, t_i, $r_{i-1}<s_{i-1}$) is independent of the precise values of s_{i-1} and r_{i-1}. The probability that $r_{i-1}<s_{i-1}$ is $p^2/(1-pq)$. Let the conditional expectation value of u_i in this case be T_1. Then we have

$$\langle u_i\rangle=\frac{p^2}{1-pq}T_1+T_2 \tag{A13}$$

where

$$T_1=\sum u_i\,\mathrm{Pr}(r_i, s_i, t_i, u_i|r_{i-1}<s_{i-1}) \tag{A14}$$

and

$$T_2=\sum u_i\mathrm{Pr}(r_i, s_i, t_i, u_i, r_{i-1}, s_{i-1}). \tag{A15}$$

In equation (A14) the sum extends over all values of $r_i<s_i$, t_i and u_i. In equation (A15) the sum extends over all values of variables satisfying $s_{i-1}\leqslant r_{i-1}$ and $r_i<s_i$.

We first calculate T_1. The corresponding configuration is shown in figure A1. We define points A, B and C to be the points immediately to the right of the points P'''_{i-1}, P_{i-1} and A respectively. Using the strategy described in the text we ensure that:

(i) the bond directly below P_{i-1} is a diode;

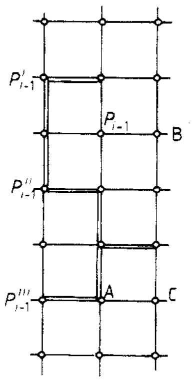

Figure A1. The calculation of T_1. The points P_{i-1}, P'_{i-1}, P''_{i-1}, P'''_{i-1}, A, B and C are shown for a typical configuration of diodes and resistors.

(ii) there is no wetting path from A to points below P'''_{i-1} using bonds in H_i and V_{i-1} alone;

(iii) there is no wetting path from B to points below P_{i-1} using bonds in H_{i-1} and V_{i-2} alone.

The *a posteriori* probabilities of bonds are modified subject to these constraints. For example, there is no uncertainty in the status of the bond below P_{i-1}. If a direct wetting path exists from B to C, then u_i must be zero in accordance with (ii) and (iii). The *a posteriori* probability of this event is easily shown to be $(pq)^{t_i+s_i-r_i}$. Let g_u be the conditional probability that $u_i = u$ given the conditions (i) and (ii) only. Then we get

$$T_1 = \left(\sum_{u=0}^{\infty} ug_u\right)\left(\sum_{s_i>r_i} \Pr(r_i)\Pr(s_i)f_{t_i}[1-(pq)^{t_i+s_i-r_i}]\right) \tag{A16}$$

which simplifies to

$$T_1 = \left(\sum_{u=0}^{\infty} ug_u\right)\frac{p^2}{1-pq}\left(1-\frac{pq^2}{1-p^2q}F(pq)\right). \tag{A17}$$

T_2 is evaluated similarly. In this case, the bond directly below P_{i-2} is known to be a diode. u_i is non-zero only if $r_i < s_i$ and $s_i + t_i - r_i > r_{i-1} - s_{i-1}$. Once the conditions are met, the conditional probability that $u_i = u$ is g_u. Writing down $P(r_i, s_i, t_i, u_i, r_{i-1}, s_{i-1})$ explicitly and summing over all the variables, subject to the constraint mentioned above, we get

$$T_2 = \left(\sum_{u=0}^{\infty} ug_u\right)\frac{p^2q}{(1-pq)^2}\left(1-\frac{q^2}{1-pq}F(q)\right). \tag{A18}$$

We now determine the probabilities g_u. Let the number of consecutive resistors in V_{i-1} below the point A be a. Then it can be shown that the probability distribution of a subject to the constraint (ii) is given by

$$\Pr(a) = (1-pq)p^aq^a. \tag{A19}$$

Let the point $(-i+1, y_{i-1}+r_i-s_i-t_i-a)$ be denoted by $A' \equiv (-i+1, a')$. By construction, the vertical bond below A′ is a diode. If a source at A′ gives the ordinate of the lowest wetted point in the column $x=-i$ equal to $a'-b$ for some $b>1$ (let us denote the corresponding conditional probability by h_b), then clearly $u_i = a+b$. Otherwise (if b is zero) u_{i+1} is zero. Thus we have

$$\sum_{u=0}^{\infty} ug_u = \left(\frac{pq}{1-pq}\right)(1-h_0) + \sum_{b=1}^{\infty} bh_b. \tag{A20}$$

The problem of determining the g_u has thus been converted to a simpler problem of determining the h_b with the simpler constraint that bonds directly below and to the left of A′ are diodes. Let the lowest point wetted by a source at A′ in the column $x=-i+1$ by using bonds in H_{i-1}, V_{i-1} and V_{i-2} be $A'' \equiv (-i+1, a'-t)$. The probability distribution of this variable t is clearly f_t. The probability that no site below P_{i-1} is wetted is q^t, the probability that none of the t bonds in H_i with ordinates lying between A′ and A″ is a resistor. Summing over t, we get

$$h_0 = \sum_{t=0}^{\infty} f_t q^t = F(q). \tag{A21}$$

If $a'-t+r$ is the ordinate of the lowest bond in H_i which allows a leftward flow $(0<r<t-1)$, then the point $(-i, a'-t+r)$ is wet. We can then wet a lower point $(-i, a'-t+r-s)$ using first-order backflows in V_i, and then a lower point $(-i, a'-t+r-s-t')$ using second-order backflow paths in V_i, V_{i-1} and H_i, and finally, a still lower point $P_i \equiv (-i, a'-t+r-s-t'-u')$ using third-order backflow paths in V_i, H_i, V_{i-1}, H_{i-1} and V_{i-2}. Obviously, the distribution of these variables s, t' and u' is the same as that of s_i, t_i and u_i, given a fixed value $r_i=r$ and $r_{i-1}<s_{i-1}$. We thus get

$$\sum_{b=1}^{\infty} bh_b = \sum (t-r+s+t'+u')f_t q^r p p^s q \Pr(t', u'|t, r, s) \tag{A22}$$

where the summation on the right-hand side extends over t varying from 1 to ∞, r varying from 0 to $t-1$, and all values of s, t' and u'. Substituting an explicit expression for $\Pr(t', u', t, r, s)$, we get

$$\sum_{b=1}^{\infty} bh_b = \frac{p^2(1-p^2q)}{q^2(1-pq)} - \left(\frac{q}{p} - \frac{p}{q}\right)[1-F(q)] + [1-F(pq)]\frac{p^2}{1-pq}\left(\frac{p^2(1-p^2q)}{q^2(1-pq)} + T_1\right). \tag{A23}$$

This equation, together with earlier equations, self-consistently determines T_1, and hence $\langle u_i \rangle$. The explicit algebraic expression for $\langle u_i \rangle$ is somewhat complicated and is omitted here.

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